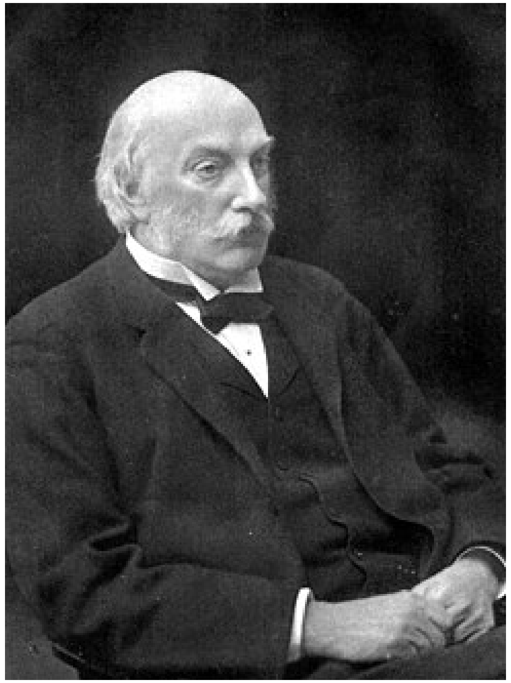
Classic Paper

On the Instability of Jets. *By* LORD RAYLEIGH, F.R.S.

[*Read November 14th*, 1878.]*



John William Strutt

Anubhav Mahapatra
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Droplets instability in everyday life!

Many more like urine droplet dynamics and inkjet printer works the same way!

Source:Wikipedia

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Numerical study of Rayleigh fission of a charged viscous liquid drop

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The Rayleigh Instability of Water Drops in the Presence of External Electric Fields

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The Rayleigh instability of small water drops is usually studied in the laboratory by allowing the drops to evaporate while supporting them in an external electric field. It is shown that, although this field is several orders of magnitude smaller than the radial field of the charged drops, it produces appreciable deviations from sphericity near the instability point even though the limiting value of charge is affected very little. These experiments are thus not suitable for verifying the theoretically predicted behavior of an isolated drop, though they probably do model quite well the behavior of droplets evaporating in the electric field of the atmosphere.

A liquid jet, initially of constant radius, is falling vertically under gravity. The liquid length increases and reaches a critical value. At this critical value, the jet loses its cylindrical shape as it decomposes into a stream of droplets. This phenomenon occurs primarily as a result of surface tension.

Joseph Plateau first characterized this instability in 1873 through experimental observation, building on the work of Savart. He noted the instability arose when the liquid column length exceeded the column diameter by a factor of about 3.13 (Plateau, 1873). Lord Rayleigh later corroborated Plateau's work, giving an analytical explanation of this physical observation.

This liquid behavior derives from the existence of small perturbations in any physical system. All real-world flows have some non-negligible external disturbance that will increase exponentially in unstable systems.

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Many, it may even be said, most of the still unexplained phenomena of Acoustics are connected with the instability of jets of fluid. For this instability there are two causes; the first is operative in the case of jets of heavy liquids, *e.g.*, water, projected into air (whose relative density is negligible), and has been investigated by Plateau in his admirable researches on the figures of a liquid mass, withdrawn from the action of gravity. It consists in the operation of the capillary force, whose effect is to render the infinite cylinder an unstable form of equilibrium, and to favour its disintegration into detached masses whose aggregate surface is less than that of the cylinder. The other cause of instability, which is operative even when the jet and its environment are of the same material, is of a more dynamical character.

With respect to instability due to capillary force, the principal problem is the determination, as far as possible, of the mode of disintegration of an infinite cylinder, and in particular of the number of masses into which a given length of cylinder may be expected to distribute itself. It must, however, be observed that this problem is not so definite as Plateau seems to think it; the mode of falling away from unstable equilibrium necessarily depends upon the peculiarities of the small displacements to which a system is subjected, and without which the position of equilibrium, however unstable, could not be departed from. Nevertheless, in practice, the latitude is not very great, because some kinds of disturbance produce their effect much more rapidly than others. In fact, if the various disturbances be represented initially by $a_1, a_2, a_3, \ldots$, and after a time t by $a_1e^{q_1t}$, $a_2e^{q_2t}$, $a_3e^{q_3t}, \ldots$, the (positive) quantities q_1, q_2, q_3, &c., being in descending order of magnitude, it is easy to see that, when $a_1, a_2, \ldots$ are small enough, the first kind necessarily acquires the preponderance. For example, at time t the ratio of the second kind to the first is $\frac{a_2}{a_1}e^{-(q_1-q_2)t}$, which, independently of the value of $a_2 : a_1$, can be made as small as we please by taking t great enough. But, in order to allow the application of the analytical expressions for so extended a time, it is generally necessary to suppose the whole amount of disturbance to be originally extremely small.†

Let us, then, taking the axis of z along the axis of the cylinder, suppose that at time t the surface of the cylinder is of the form

$$r = a + \alpha \cos \kappa z \quad \text{.............................(1)};$$

where α is a small quantity variable with the time, and $\kappa = 2\pi\lambda^{-1}$, λ being the *wave-length* of the original disturbance. The information that we require will be readily obtained by Lagrange's method, when we have calculated expressions for the potential and kinetic energies of the motion represented by (1).

The potential energy due to the capillary forces is a question merely of the surface of the liquid. If we denote the surface corresponding (on the average) to the unit length along the axis by σ, we readily find

$$\sigma = 2\pi a + \tfrac{1}{2}\pi a \kappa^2 \alpha^2 \quad \text{.............................(2)}.$$

In this, however, we have to substitute for a (which is not strictly constant) its value obtained from the condition that S, the volume enclosed per unit of length, is given. We have

$$S = \pi a^2 + \tfrac{1}{2}\pi \alpha^2 \quad \text{.............................(3)},$$

whence $$a = \sqrt{\left(\frac{S}{\pi}\right)} \cdot \left(1 - \tfrac{1}{4}\frac{\pi\alpha^2}{S}\right) \quad \text{.......................(4)}.$$

Using this in (2), we get with sufficient approximation

$$\sigma = 2\sqrt{(\pi S)} + \frac{\pi\alpha^2}{2a}(\kappa^2 a^2 - 1) \quad \text{.......................(5)};$$

or, if σ_0 be the value of σ for the undisturbed condition,

$$\sigma - \sigma_0 = \frac{\pi\alpha^2}{2a}(\kappa^2 a^2 - 1) \quad \text{.......................(6)}.$$

From this we infer that, if $\kappa a > 1$, the surface is greater after displacement than before; so that, if $\lambda < 2\pi a$, the displacement is of such a character that with respect to it the system is stable. We are here concerned only with values of κa less than unity. If T_1 denote the cohesive tension, the potential energy V reckoned per unit of length from the position of equilibrium is

$$V = -T_1 \frac{\pi\alpha^2}{2a}(1 - \kappa^2 a^2) \quad \text{.......................(7)}.$$

We have now to calculate the kinetic energy of motion. It is easy to prove that the velocity potential is of the form

$$\phi = A J_0(i\kappa r) \cos \kappa z \quad \text{...........................(8)},$$

J_0 being the symbol of Bessel's functions of zero order, so that

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \frac{x^6}{2^2 . 4^2 . 6^2} + \ldots\ldots \quad \text{.........(9)}.$$

The coefficient A is to be determined from the consideration that the outwards normal velocity at the surface of the cylinder is equal to $\dot{\alpha}\cos\kappa z$. Hence

$$i\kappa A\, J_0'(i\kappa a) = \dot{\alpha} \quad\text{.........................(10).}$$

Denoting the density by ρ, we have for the kinetic energy the expression

$$T = \tfrac{1}{2}\rho \int 2\pi a\,.\,\phi\, \frac{d\phi}{dr_{(=a)}}\, dz\,;$$

or, if we reckon it in the same way as V per unit of length,

$$T = \tfrac{1}{2}\rho\,\pi\, a^2\, \frac{J_0(i\kappa a)\,\dot{\alpha}^2}{i\kappa a\, J_0'(i\kappa a)} \quad\text{.........................(11).}$$

Thus, by Lagrange's method, if $\alpha \propto e^{qt}$,

$$q^2 = \frac{T_1}{\rho a^3}\, \frac{(1-\kappa^2 a^2)\,.\, i\kappa a\,.\, J_0'(i\kappa a)}{J_0(i\kappa a)} \quad\text{.........................(12),}$$

which determines the law of falling away from equilibrium for a disturbance of wave-length λ. The solutions for the various values of λ and the corresponding energies are independent of one another; and thus, by Fourier's theorem, it is possible to express the condition of the system at time t, after the communication of any infinitely small disturbances symmetrical about the axis. But what we are most concerned with at present is the value of q^2 as a function of κa, and especially the determination of that value of κa for which q^2 is a maximum. That such a maximum must exist is evident *a priori*. Writing x for κa, we have to examine the values of

$$\frac{(1-x^2)\,.\, ix\,.\, J_0'(ix)}{J_0(ix)} \quad\text{.........................(13).}$$

Expanding in powers of x, we may write, for (13),

$$\tfrac{1}{2}x^2(1-x^2)\left\{1 - \frac{x^2}{2^3} + \frac{x^4}{2^4\,.\,3} - \frac{11x^6}{2^{10}\,.\,3} + \frac{19x^8}{2^{11}\,.\,3\,.\,5} + \dots\right\} \quad\text{.....(14).}$$

or

$$\tfrac{1}{2}\left\{x^2 - \tfrac{9}{8}x^4 + \frac{7}{2^4\,.\,3}x^6 - \frac{25}{2^{10}}x^8 + \frac{91}{2^{11}\,.\,3\,.\,5}x^{10} + \dots\right\} \quad\text{......(15).}$$

Hence, to find the maximum, we obtain by differentiation

$$1 - \tfrac{9}{4}x^2 + \frac{7}{2^4}x^4 - \frac{100}{2^{10}}x^6 + \frac{91}{2^{11}\,.\,3}x^8 + \dots = 0 \quad\text{...........(16).}$$

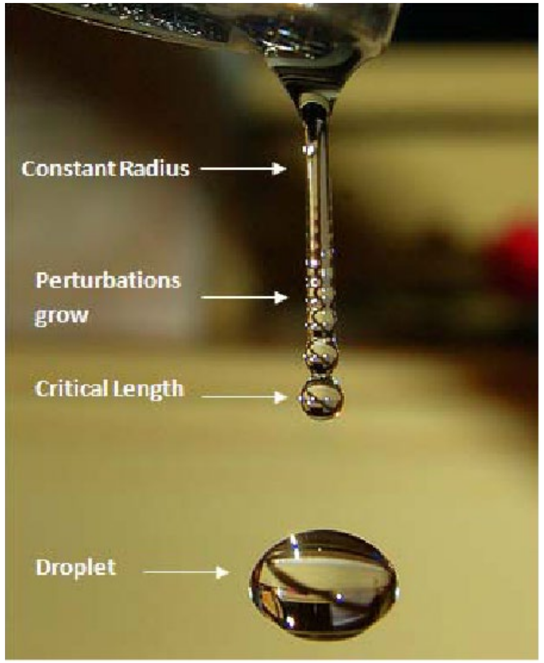
If the last two terms be neglected, the quadratic gives $x^2 = {\cdot}4914$. If this value be substituted in the small terms, the equation becomes

$$\cdot 98928 - \tfrac{9}{4}x^2 + \tfrac{7}{16}x^4 = 0,$$

whence

$$x^2 = {\cdot}4858 \quad\text{.................................(17).}$$

Rayleigh predicted that a charged conducting liquid drop of radius **a** develops instabilities and breaks up when its charge **Q** exceeds the limit where Q_R is the critical charge **γ, ε** is the surface tension and dielectric constant of the liquid, respectively, and ε_0 is the permittivity of free space.



Picture of instability

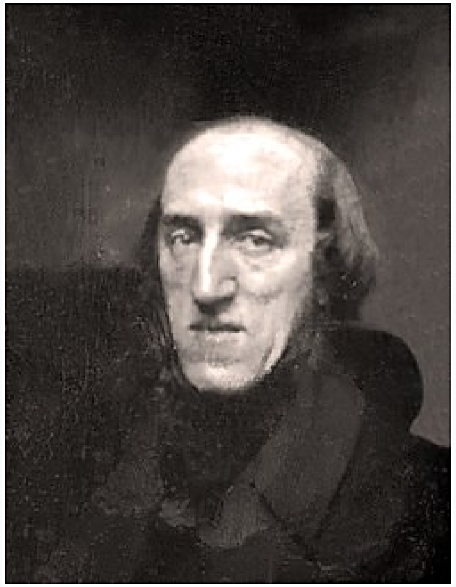
$$Q_R = 8\pi\sqrt{\epsilon\epsilon_0\gamma a^3}$$

$Q >> Q_R$

Rayleigh limit

Biography

Joseph Plateau

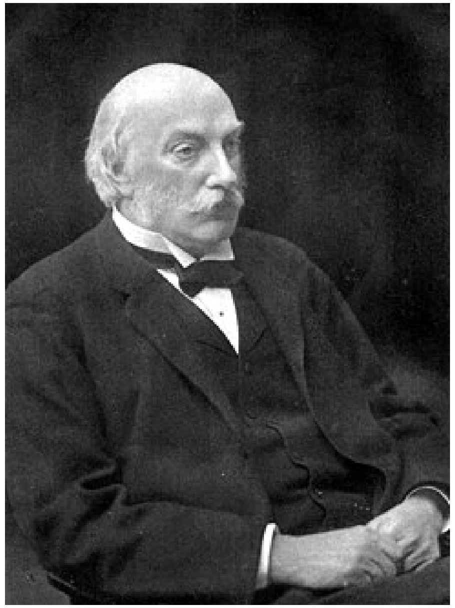


Plateau in 1843

Born	14 October 1801[1] Brussels, French Republic
Died	15 September 1883 (aged 81) Ghent,[2] Belgium
Nationality	Belgian
Alma mater	University of Liège
Known for	Physics of soap bubbles (Plateau's laws), Plateau's problem
	Scientific career
Institutions	Ghent University
Doctoral advisor	Adolphe Quetelet

-Thank you

The Lord Rayleigh



Born	12 November 1842 Maldon, Essex, England
Died	30 June 1919 (aged 76) Terling Place, Witham, Essex, England
Alma mater	Trinity College, Cambridge (BA, 1865; MA, 1868)
Known for	Discovering argon (1894) *See list*
Spouse	Evelyn Balfour (m. 1871)
Children	3, including Robert
Awards	Smith's Prize (1865) FRS (1873) Royal Medal (1882) De Morgan Medal (1890) Matteucci Medal (1895) Barnard Medal for Meritorious Service to Science (1895) Faraday Lectureship Prize (1895) Copley Medal (1899) Nobel Prize in Physics (1904) Albert Medal (1905) Elliott Cresson Medal (1913) Rumford Medal (1914)
Honours	Order of Merit (1902)
	Scientific career
Fields	Physics
Institutions	Cavendish Laboratory (1879–1884) Royal Institution (1887–1905)
Academic advisors	Lord Kelvin[1] Edward John Routh George Stokes[2]
Notable students	Jagadish Chandra Bose William Ramsay J. J. Thomson